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A SUPERCONVERGENT DISCONTINUOUS GALERKIN METHOD FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS, SMOOTH AND NON-SMOOTH KERNELS

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Abstract. We study the numerical solution for Volerra integro-differential equations with smooth and non-smooth kernels. We use a h-version discontinuous Galerkin (DG) method and derive nodal error bounds that are explicit in the parameters of interest. In the case of non-smooth kernel, it is justified that the start-up singularities can be resolved at superconvergence rates by using non-uniformly graded meshes. Our theoretical results are numerically validated in a sample of test problems.

Integro-differential equation, weakly singular kernel, smooth kernel, DG time-stepping, error analysis, variable time steps

1. Introduction

In this paper, we study the discontinuous Galerkin (DG) for a nonlocal time dependent Volterra integro-differential equation of the form

$$u'(t) + a(t)u(t) + \mathcal{B}u(t) = f(t),\ 0 < t < T \text{ with } u(0) = u_0, \tag{1.1}$$

where $\mathcal{B}$ is the Volterra operator:

$$\mathcal{B}u(t) = \int_0^t \beta(t,s)u(s)\,ds, \tag{1.2}$$

such that,

$$\beta(t,s) = (t-s)^{\alpha-1}b(s) \quad \text{for all } 0 < s < t \le T \tag{1.3}$$

with either $\alpha \in (0,1)$ (weakly singular kernel) or $\alpha \in \mathbb{N}_0 := \{1, 2, 3, \cdots\}$ (smooth kernel). Here a, b and f are continuous real valued functions on $[0,T]$. We assume that there exist $\mu_* > 0$ such that $a(t) \ge \mu_*$ for all $t \in [0,T]$. As a consequence of this and the continuity assumptions on the functions a and b; there exist $\mu_*, \mu^* > 0$ such that

$$\mu_* \le a(t) \le \mu^* \quad \text{and} \quad |b(t)| \le \mu^* \quad \text{for all } t \in [0,T]\,. \tag{1.4}$$

For any $u_0 \in R$, problem (1.1) has a unique solution u which is continuously differentiable, see for example [1]. However for $\alpha \in (0,1)$, even if the functions a, b and f in (1.1)–(1.3) are smooth, the second derivative of u is not bounded at $t = 0$ (see [3] and related references therein), and behaves like $|u''(t)| \le Ct^{\alpha-1}$. The singular behavior of u near $t = 0$ may lead to suboptimal convergence rates if

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we work with quasi-uniform time meshes. To overcome this problem, we employ a family of non-uniform meshes, where the time-steps are concentrated near $t = 0$.

Various numerical methods had been studied for problem (1.1). For instance, collocation methods for (1.1) with a weakly singular kernel were investigated by many authors where an $O(k^{p+1})$ (k is the maximum time-step size and p is the degree of the approximate solution) global convergence rate had been achieved using a non-uniform graded mesh of the form (2.10), see for example [1, 3, 21] and references therein. Spectral methods and the corresponding error analysis were provided in [7, 22] assuming that $\alpha = 1$ and the solution u of (1.1) is smooth. However, for $0 < \alpha < 1$ (that is, the kernel is weakly singular), the spectral collocation method were recently studied in [23] where the convergence analysis was carried out assuming again that the solution u is smooth. For other numerical tools, refer to [23] and references therein.

In the present paper we shall study the nodal error analysis for the DG time-stepping method (with a fixed approximation order) applied to problem (1.1). Indeed, the DG time-stepping method for (1.1) when $\alpha \in (0, 1)$ has been introduced in [2], where a uniform optimal $O(k^{p+1})$ convergence rate had been shown assuming that u is sufficiently regular. In this work, we show that a faster convergence than $O(k^{p+1})$ is possible at the nodal points. For a weakly singular kernel ($\alpha \in (0, 1)$), we prove that by using non-uniformly refined time-steps, start-up singularities near $t = 0$ can be resolved at $O(k^{\min\{p,\alpha+1\}+p+1})$ superconvergence rates. Such convergence rates can not be obtained by using the approach given in [2]. Very briefly, our proof technique will be carried out in two steps; deriving first the global convergence results of the DG method for the dual problem of (1.1) (which is essential for the nodal error but irrelevant for the global error estimates), see Theorem 4.1. Then, we use these results with the orthogonal property of the DG scheme for (1.1) very appropriately (see (5.1) and Theorem 5.1) to achieve nodal superconvergence estimates. For smooth kernels ($\alpha \in \mathbb{N}_0$), we appropriately modify our earlier analyses to show nodal superconvergence rates of order $O(k^{2p+1})$ assuming that the functions a, b and f are sufficiently regular (see Theorem 6.2).

The origins of the DG methods can be traced back to the seventies where they had been proposed as variational methods for numerically solving initial-value problems and transport problems [10, 18, 4, 6, 8] and the references therein. In the eighties, DG time-stepping methods were successfully applied to parabolic problems; see for example, [5], where a nodal $O(k^{2p+1})$ superconvergence rate had been proved. Subsequently, in [9], a piecewise linear time-stepping DG method had been proposed and studied for a parabolic integro-differential equation:

$$u_t + Au + \mathcal{B}\widetilde{A}u = f \text{ in } (0, T] \times \Omega \text{ with } u(0) = v(x) \text{ on } \Omega \text{ for } \alpha \in (0, 1), \tag{1.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded convex domain, A is a linear self-adjoint, positive-definite operator (spatial), with compact inverse, defined in $D(A)$, and where A dominates the spatial operator $\widetilde{A}$. A nodal $O(k^3)$ superconvergence rate had been derived assuming that $b(s) = 1$ in (1.3), where the error analysis there was based on the fact that on each time interval, the DG solution takes its maximum values on one of the end points. However, this is not true in the case of DG methods of higher order p. The high order time-stepping DG for (1.5) was investigated in [15] where a global optimal $O(k^{p+1})$ convergence rate had been proved, assuming that the mesh is non-uniformly graded. (For other numerical methods for (1.5), see [12, 14, 16, 17]

and related references therein.) Indeed, our convergence analysis can in principle be extended to cover the nodal error estimates from the DG time-stepping method of order p, applied to (1.5).

The outline of the paper is as follows. In Section 2, we introduce the DG time-stepping method with a fixed approximation degree p (typically low) on non-uniformly refined time-steps with $p \ge 1$. In Section 3, we give a global formulation of the DG scheme, introduce our projection operator, and also provide some technical lemmas. In Section 4, we define the dual of the problem (1.1) and then derive the error estimates from the discretization by the DG method when $\alpha \in (0,1)$; see Theorem 4.1. In Section 5, we prove our main nodal error bounds. For $\alpha \in (0,1)$, an error $|U_-^n - u(t_n)|$ of order $O(k^{\min\{p,\alpha+1\}+p+1})$ (i.e., superconvergent of order k^3 for $p=1$ and $k^{p+2+\alpha}$ for $p \ge 2$) has been shown provided that the solution u of (1.1) satisfies (2.7) and the mesh grading parameter $\gamma > (p+1)/\sigma$; see Theorem 5.1. In Section 6, we consider the case $\alpha \in \mathbb{N}_0$ (in (1.3)) and thus the kernel is smooth. We show a nodal error of order $O(k^{2p+1})$ (over a uniform mesh) assuming that the solution u of (1.1) is sufficiently regular, refer to Theorem 6.2. We present a series of numerical examples to validate our theoretical results in Section 7.

2. Discontinuous Galerkin time-stepping

To describe the DG method, we introduce a (possibly non-uniform) partition of the time interval $[0,T]$ given by the points

$$0 = t_0 < t_1 < \cdots < t_N = T. \tag{2.1}$$

We set $I_n = (t_{n-1}, t_n)$ and $k_n = t_n - t_{n-1}$ for $1 \le n \le N$. The maximum step-size is defined as $k = \max_{1\le n\le N} k_n$. We now introduce the discontinuous finite element space

$$\mathcal{W}_p = \{\, v : J_N \to \mathbb{R} \,:\, v|_{I_n} \in \mathbb{P}_p,\ 1 \le n \le N \}\,, \tag{2.2}$$

where $J_N = \cup_{n=1}^N I_n$, and $\mathbb{P}_p$ denotes the space of polynomials of degree $\le p$ where p is a positive integer ≥ 1. We denote the left-hand limit, right-hand limit and jump at t_n by $v_-^n = v(t_n^-)$, $v_+^n = v(t_n^+)$ and $[v]^n = v_+^n - v_-^n$, respectively.

The DG approximation $U \in \mathcal{W}_p$ is now obtained as follows: Given $U(t)$ for $t \in I_j$ with $1 \le j \le n-1$, the approximation $U \in \mathbb{P}_p$ on the next time-step I_n is determined by requesting that

(2.3)

$$U_+^{n-1} X_+^{n-1} + \int_{t_{n-1}}^{t_n} \Big[U' + a(t)U(t) + \mathcal{B}U(t)\Big] X\, dt = U_-^{n-1} X_+^{n-1} + \int_{t_{n-1}}^{t_n} f\, X\, dt$$

for all test functions $X \in \mathbb{P}_p$. This time-stepping procedure starts from $U_-^0 = u_0$, and after N steps it yields the approximate solution $U \in \mathcal{W}_p$ for $t \in J_N$.

Remark 2.1. For the piecewise-constant case $p = 0$, since $U'(t) = 0$ and $U(t) = U_-^n = U_+^{n-1} =: \mathbf{U}^n$ for $t \in I_n$, the DG method (2.3) amounts to a generalized

backward-Euler scheme

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k_n} + U^n \frac{1}{k_n}\int_{t_{n-1}}^{t_n} a(t)\,dt + \omega_{nn} k_n \mathbf{U}^n$$
$$= \frac{1}{k_n}\int_{t_{n-1}}^{t_n} f(t)\,dt - \frac{1}{k_n}\int_{t_{n-1}}^{t_n} \sum_{j=1}^{n-1} \mathbf{U}^j \int_{t_{j-1}}^{\min(t,t_j)} (t-s)^{\alpha-1} b(s)\,ds\,dt\,.$$

In this case, the nodal and global errors have the same rate of convergence which is $O(k)$, see [2, Theorem 3.8].

For our error analysis, it will be convenient to reformulate the DG scheme (2.3) in terms of the global bilinear form

(2.4)

$$G_N(U,X) = U_+^0\, X_+^0 + \sum_{n=1}^{N-1} [U]^n\, X_+^n + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Big[U'(t) + a(t)U(t) + \mathcal{B}U(t)\Big] X\, dt.$$

By summing up (2.3) over all the time-steps and using $U_-^0 = u_0$, the DG method can now equivalently be written as: Find $U \in \mathcal{W}_p$ such that

$$G_N(U,X) = u_0\, X_+^0 + \int_0^{t_N} f\, X\, dt \quad \forall\; X \in \mathcal{W}_p. \tag{2.5}$$

Since the solution u is continuous, it follows that

$$G_N(u,X) = u_0\, X_+^0 + \int_0^{t_N} f\, X\, dt \quad \forall\; X \in \mathcal{W}_p.$$

Thus, the following Galerkin orthogonality property holds:

$$G_N(U-u,X) = 0 \quad \forall\; X \in \mathcal{W}_p\,. \tag{2.6}$$

Before stating the regularity property of the solution u of (1.1), we display in the next remark an alternative form of G_N which will be used in our error analysis.

Remark 2.2. Integration by parts yields the following alternative expression for the bilinear form G_N in (2.4):

$$G_N(U,X) = U_-^N\, X_-^N - \sum_{n=1}^{N-1} U_-^n\, [X]^n$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[-U(t)X' + a(t)U(t)X + \mathcal{B}U(t)X\right] dt.$$

Throughout the paper, we assume that the solution u of (1.1) satisfies:

$$|u^{(j)}(t)| \le C\, t^{\sigma-j} \quad \text{for} \quad 1 \le j \le p+1 \text{ where } 1 \le \sigma \le \alpha+1 \tag{2.7}$$

where the constant C depends on j. For instance, if in (1.1) the function $f = t^{\kappa_1} f_1 + t^{\kappa_2} f_2$ for some κ_1, $\kappa_2 \ge 0$ and the functions a, b, f_1 and f_2 are in $\mathrm{C}^{j-1}[0,T]$ for $1 \le j \le p$, then (2.7) holds for $\sigma = 1 + \min\{\kappa_1, \kappa_2, \alpha\}$, see [1, Section 7.1] for more details.

We notice from (2.7) that $|u^{(j)}(t)|$ is not bounded near $t = 0$ for $j \ge 2$. Hence, to compensate the singular behavior of u near $t = 0$, we employ a family of non-uniform meshes, where the time-steps are concentrated near zero. Thus, we assume

that, for a fixed $\gamma \geq 1$,

$$k_n \leq C_\gamma k t_n^{1-1/\gamma} \quad \text{and} \quad t_n \leq C_\gamma t_{n-1} \quad \text{for } 2 \leq n \leq N, \tag{2.8}$$

with

$$c_\gamma k^\gamma \leq k_1 \leq C_\gamma k^\gamma. \tag{2.9}$$

For instance, one may choose

$$t_n = (n/N)^\gamma T \quad \text{for } 0 \leq n \leq N. \tag{2.10}$$

Under the assumptions (2.7)–(2.9), we show in Theorem 5.1 that the error $|U_-^n - u(t_n)|$ is of order $k^{\gamma\sigma+\min\{p,1+\alpha\}}$, for $1 \leq n \leq N$. So, we have a superconvergence of order $k^{p+1+\min\{p,1+\alpha\}}$ provided $\gamma > (p+1)/\sigma$. However, for a quasi-uniform mesh (i.e., $\gamma = 1$) our bound yields a poorer convergence rate of order $k^{\sigma+\min\{p,1+\alpha\}}$.

3. Projection operator and technical lemmas

In this section we introduce a projection operator that has been used various times in the analysis of DG time-stepping methods; see [24], and state some preliminary results that are needed in our convergence analysis in the forthcoming sections.

For a given function $u \in C[0,T]$, we define the interpolant $\Pi^- u \in \mathcal{W}_p$ by

$$\Pi^- u(t_n^-) = u(t_n) \text{ and } \int_{t_{n-1}}^{t_n} (u - \Pi^- u)\, v\, dt = 0 \quad \forall\, v \in \mathbb{P}_{p-1}(I_n) \tag{3.1}$$

and for $1 \leq n \leq N$. From [19, Lemma 3.2] it follows that Π^- is well-defined.

To state the approximation properties of Π^-, we introduce the notation

$$\|\phi\|_{I_n} = \sup_{t \in I_n} |\phi(t)| \quad \text{for any } \phi \in C(t_{n-1}, t_n).$$

Theorem 3.1. *There exists a constant C, which depends on p such that:*

(i) For any $0 \leq q \leq p$ and $u|_{I_n} \in H^{q+1}(I_n)$, there holds

$$\int_{t_{n-1}}^{t_n} |\Pi^- u - u|^2\, dt \leq C k_n^{2q+2} \int_{t_{n-1}}^{t_n} |u^{(q+1)}|^2\, dt \quad \text{for } 1 \leq n \leq N.$$

(ii) For any $0 \leq q \leq p$ and $u|_{I_n} \in H^{q+1}(I_n) \cap C(I_n)$, there holds

$$\|\Pi^- u - u\|_{I_n}^2 \leq C k_n^{2q+1} \int_{t_{n-1}}^{t_n} |u^{(q+1)}|^2\, dt \quad \text{for } 1 \leq n \leq N.$$

Proof. For the proof of the first bound, we refer to [19, Section 3] or [24, Chapter 12, Page 214]. For the second bound, see [20, Theorem 3.9 and Corollary 3.10] or [24, Equation (12.10)]. ∎

The following two technical lemmas are needed in our derivation of the error estimates. The first lemma has been proved in [9, Lemma 6.3].

Lemma 3.2. *If $g \in L_2(0,T)$ and $\alpha \in (0,1)$ then*

$$\int_0^T \left(\int_0^t (t-s)^{\alpha-1} g(s)\, ds \right)^2 dt \leq \frac{T^\alpha}{\alpha} \int_0^T (T-t)^{\alpha-1} \int_0^t g^2(s)\, ds\, dt.$$

The next lemma is the following Gronwall inequality; see [9, Lemma 6.4].

Lemma 3.3. *Let* $\{a_j\}_{j=1}^N$ *and* $\{b_j\}_{j=1}^N$ *be sequences of non-negative numbers with* $0 \le b_1 \le b_2 \le \cdots \le b_N$. *Assume that there exists a constant* $K \ge 0$ *such that*

$$a_n \le b_n + K \sum_{j=1}^{n} a_j \int_{t_{j-1}}^{t_j} (t_n - t)^{\alpha-1}\, dt \quad \text{for } 1 \le n \le N \text{ and } \alpha \in (0,1).$$

Assume further that $\delta = \frac{K\,k^\alpha}{\alpha} < 1$. *Then for* $n = 1, \cdots, N$, *we have* $a_n \le C b_n$ *where* C *is a constant that solely depends on* K, T, α *and* δ.

Throughout the rest of the paper, we shall always implicitly assume that the *maximum step-size* k *is sufficiently small* so that the condition $\delta < 1$ in Lemma 3.3 is satisfied. More precisely, following Lemma 4.2, we shall require that

$$4\,T^\alpha \left(\frac{\mu^*}{\alpha\,\mu_*}\right)^2 k^\alpha < 1\,.$$

4. Error analysis of the dual problem

This section is devoted to deriving error estimates for the DG method applied to the dual problem of the Volterra integro-differential equation (1.1). The main results of this section (more precisely, Theorem 4.1) play a crucial role in the proof of the superconvergence error estimate in section 5.

Let z be the solution of the dual problem

$$-\,z' + a(t)z(t) + \mathcal{B}^* z(t) = 0 \quad \text{for } 0 \le t < T, \quad \text{with } z(T) = z_T, \tag{4.1}$$

where $\mathcal{B}^* v(t) = \int_t^T \beta(s,t) v(s)\, ds$ ($\mathcal{B}^*$ is the dual of the integral operator $\mathcal{B}$).

Since z has no jumps and since

$$\int_0^T \big[-v(t)z'(t) + a(t)v(t)z(t) + \mathcal{B}v(t)\, z(t)\big]\, dt$$
$$= \int_0^T v(t)(-z'(t) + a(t)z(t) + \mathcal{B}^* z(t))\, dt = 0,$$

the alternative expression of G_N given in Remark 2.2 yields the identity

$$G_N(v, z) = v_-^N z_T \quad \text{forall } v \in C[0,T]\,. \tag{4.2}$$

($C(0,T]$ denotes the space of continuous functions on $[0,T]$). Let $Z \in \mathcal{W}_p$ denote the approximate solution of (4.1) given by

$$G_N(V, Z) = V_-^N z_T \quad \forall\ V \in \mathcal{W}_p\,. \tag{4.3}$$

Hence, the following Galerkin orthogonality property holds:

$$G_N(V, Z - z) = 0 \quad \forall\ V \in \mathcal{W}_p\,. \tag{4.4}$$

At this stage, the main aim is to estimate the error $Z - z$ in the L_2-norm. First it is good to notice that (4.4) is a discrete backward analogue of (2.6). Since it is more convenient to deal with a discrete forward problem, we introduce the functions $\tilde{z}(t) = z(t_N - t)$ and $\tilde{Z}(t) = Z(t_N - t)$ and then, (4.4) can be rewritten as;

$$\tilde{G}_N(\tilde{Z} - \tilde{z}, V) = 0 \quad \forall\ V \in \widetilde{\mathcal{W}}_p; \tag{4.5}$$

where $\tilde{G}_N$ is defined as in (2.4) but with $\tilde{a}(t) := a(t_N - t)$ in place of $a(t)$ and $\beta(t_N - s, t_N - t)$ in place of $\beta(t,s)$. The finite dimensional space $\widetilde{\mathcal{W}}_p$ is defined as

$\mathcal{W}_p$ but on the reverse mesh: $0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_N$, where $\tilde{t}_i = \tilde{t}_{i-1} + \tilde{k}_i$ with $\tilde{k}_i = k_{N+1-i}$.

Setting $\zeta = \tilde{\Pi}^- \tilde{z} - \tilde{z}$ and $\theta = \tilde{Z} - \tilde{\Pi}^- \tilde{z}$ where $\tilde{\Pi}^-$ is the interpolant operator defined as in (3.1), but on the reverse mesh. Then (4.5) implies that

$$\tilde{G}_N(\theta, V) = -\tilde{G}_N(\zeta, V) \quad \forall\, V \in \mathcal{W}_p\,. \tag{4.6}$$

By the construction of the interpolant we have $\zeta(\tilde{t}^n_-) = 0$ for all $n \geq 1$ and hence, using the alternative expression for G_N given in Remark 2.2 and $\int_{\tilde{t}_{n-1}}^{\tilde{t}_n} \zeta(t)\, V'(t)\, dt = 0$ (by definition of the operator Π^-),

$$\tilde{G}_N(\zeta, V) = \sum_{n=1}^{N} \int_{\tilde{t}_{n-1}}^{\tilde{t}_n} \big[\tilde{a}(t)\zeta(t)V(t) + \tilde{\mathcal{B}}\zeta(t)V(t)\big]\, dt \tag{4.7}$$

where

$$\tilde{\mathcal{B}}\zeta(t) = \int_0^t \beta(t_N - s, t_N - t)\zeta(s)\, ds\,.$$

In the next theorem we estimate the error between z and Z.

Theorem 4.1. *If z is the solution of the backward VIE* (4.1)*, and if $Z \in \mathcal{W}_p$ is the approximate solution defined by* (4.3)*, then*

$$\int_0^{t_N} |z - Z|^2\, dt \leq C k^{2\alpha+2} |z_T|^2$$

provided that

$$\int_0^{t_N} |\theta(t)|^2\, dt \leq C \int_0^{t_N} |\zeta(t)|^2\, dt\,. \tag{4.8}$$

Proof. From the decomposition: $\tilde{Z} - \tilde{z} = \zeta + \theta$, the triangle inequality, and (4.8), we have

$$\int_0^{t_N} |z - Z|^2\, dt = \int_0^{t_N} |\tilde{z} - \tilde{Z}|^2\, dt \leq C \int_0^{t_N} |\zeta|^2\, dt. \tag{4.9}$$

Thus, the task reduces to bound the right-hand side of (4.9). Starting from the relation $\tilde{z}(t) = z(t_N - t)$ and recalling that z satisfies (4.1), it is clear that $\tilde{z}$ solves the VIE:

$$\tilde{z}' + a(t_N - t)\tilde{z}(t) + \int_0^t \beta(t_N - s, t_N - t)\tilde{z}(s)\, ds = 0 \quad \text{for } 0 < t < T,$$

with $\tilde{z}(0) = z_T$. Hence, an application of (2.7) for $\sigma = \alpha + 1$ with $\tilde{z}$ in place of u gives

$$|\tilde{z}'(t)| + t^{1-\alpha}|\tilde{z}''(t)| + t^{2-\alpha}|\tilde{z}'''(t)| \leq C|z_T|\,. \tag{4.10}$$

Now, Theorem 3.1 on the reverse mesh (with ζ in place of $\Pi^- u - u$) and (4.10) yield

(4.11)
$$\begin{aligned}\sum_{n=2}^{N}\int_{\tilde t_{n-1}}^{\tilde t_n}|\zeta(t)|^2\,dt &\le C\sum_{n=2}^{N}\tilde k_n^4\int_{\tilde t_{n-1}}^{\tilde t_n}|\tilde z''(t)|^2\,dt \le C\sum_{n=2}^{N}\tilde k_n^4\int_{\tilde t_{n-1}}^{\tilde t_n}t^{2\alpha-2}|z_T|^2\,dt\\ &\le C|z_T|^2\sum_{n=2}^{N}\tilde k_n^5\tilde t_{n-1}^{2\alpha-2} = C|z_T|^2\sum_{n=2}^{N}\tilde k_n^{3+2\alpha}(\tilde k_n/\tilde t_{n-1})^{2-2\alpha}\\ &\le C|z_T|^2\sum_{n=2}^{N}\tilde k_n^{3+2\alpha} \le Ck^{2\alpha+2}|z_T|^2\end{aligned}$$

and on $(0,\tilde t_1)$, we notice for $1/2<\alpha\le 1$ that

$$\int_0^{\tilde t_1}|\zeta(t)|^2\,dt \le C\tilde k_1^4\int_0^{\tilde t_1}|\tilde z''(t)|^2\,dt \le Ck_N^4\int_0^{\tilde t_1}t^{2\alpha-2}|z_T|^2\,dt \le C|z_T|^2k_N^{3+2\alpha}\,,$$

and for $0<\alpha\le 1/2$ that

$$\begin{aligned}\int_0^{\tilde t_1}|\zeta(t)|^2\,dt \le C\tilde k_1^2\int_0^{\tilde t_1}|\tilde z'(t)|^2\,dt &\le Ck_N^2\int_0^{\tilde t_1}|z_T|^2\,dt\\ &\le C|z_T|^2k_N^3 \le C|z_T|^2k_N^{2+2\alpha}\,.\end{aligned}\tag{4.12}$$

Finally, combine (4.9) and (4.11)–(4.12), we obtain the desired result. □

In the next lemma we prove the applicability of the assumption (4.8).

Lemma 4.2. *For $1\le n\le N$, we have*

$$\int_0^{\tilde t_n}|\theta(t)|^2\,dt \le C\int_0^{\tilde t_n}|\zeta(t)|^2\,dt$$

Proof. Choosing $V=\theta$ on $(0,\tilde t_n)$ and zero elsewhere in (4.6) and (4.7), then using the alternative definition of G_N in Remark 2.2 and $\theta'\theta=(d/dt)|\theta|^2/2$, we observe that

$$\begin{aligned}|\theta(\tilde t_-^n)|^2 + |\theta(\tilde t_0^+)|^2 + \sum_{j=1}^{n-1}|[\theta]^j|^2 + 2\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt&\\ = -2\int_0^{\tilde t_n}\Big[\tilde a(t)\zeta(t)+\tilde B\zeta(t)+\tilde{\mathcal B}\theta(t)\Big]\theta(t)\,dt.&\end{aligned}$$

So

$$\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt \le \int_0^{\tilde t_n}\tilde a(t)|\theta(t)||\zeta(t)|\,dt + \int_0^{\tilde t_n}|\tilde{\mathcal B}\zeta(t)+\tilde{\mathcal B}\theta(t)|\,|\theta(t)|dt.$$

We use the geometric-arithmetic mean inequality $|xy|\le\frac{\varepsilon x^2}{2}+\frac{y^2}{2\varepsilon}$ (valid for any $\varepsilon>0$) we find that

$$\begin{aligned}\int_0^{\tilde t_n}\tilde a(t)|\theta(t)||\zeta(t)|\,dt &\le \sqrt{\mu^*}\int_0^{\tilde t_n}\sqrt{\tilde a(t)}|\theta(t)||\zeta(t)|\,dt\\ &\le \frac14\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt + \mu^*\int_0^{\tilde t_n}|\zeta(t)|^2\,dt\end{aligned}$$

and thus

$$\frac{3}{4}\int_0^{\tilde t_n} \tilde a(t)|\theta(t)|^2\,dt \;\le\; \mu^*\int_0^{\tilde t_n} |\zeta(t)|^2\,dt + \int_0^{\tilde t_n} |\tilde{\mathcal{B}}\zeta(t) + \tilde{\mathcal{B}}\theta(t)|\,|\theta(t)|dt. \tag{4.13}$$

We employ the Cauchy-Schwarz inequality, again the geometric-arithmetic mean inequality, and Lemma 3.2 (with $T = \tilde t_n$):

$$\begin{aligned}
\int_0^{\tilde t_n} |\mathcal{B}\zeta(t)\,\theta(t)|dt &\le \mu^*\int_0^{\tilde t_n}\int_0^t (t-s)^{\alpha-1}|\zeta(s)|\,|\theta(t)|\,ds\,dt\\
&\le \frac{\mu^*}{\mu_*}\int_0^{\tilde t_n} \tilde a(t)^{1/2}|\theta(t)|\int_0^t (t-s)^{\alpha-1}\tilde a(s)^{1/2}|\zeta(s)|\,ds\,dt\\
&\le \frac{\mu^*}{\mu_*}\left(\int_0^{\tilde t_n}\left(\int_0^t (t-s)^{\alpha-1}\tilde a(s)^{1/2}|\zeta(s)|\,ds\right)^2 dt\right)^{1/2}\left(\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt\right)^{1/2}\\
&\le \left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{\tilde t_n}\left(\int_0^t (t-s)^{\alpha-1}\tilde a(s)^{1/2}|\zeta(s)|\,ds\right)^2 dt + \frac{1}{4}\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt\\
&\le \frac{\tilde t_n^\alpha}{\alpha}\left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{\tilde t_n}(\tilde t_n - t)^{\alpha-1}\int_0^t \tilde a(s)|\zeta(s)|^2\,ds\,dt + \frac{1}{4}\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt\\
&\le \left(\frac{\tilde t_n^\alpha\mu^*}{\alpha\,\mu_*}\right)^2\int_0^{\tilde t_n}\tilde a(s)|\zeta(s)|^2\,ds + \frac{1}{4}\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt\,.
\end{aligned}$$

Similarly, we notice that

$$\begin{aligned}
&\int_0^{\tilde t_n} |\mathcal{B}\theta(t)\,\theta(t)|dt\\
&\qquad\le \frac{\tilde t_n^\alpha}{\alpha}\left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{\tilde t_n}(\tilde t_n - t)^{\alpha-1}\int_0^t \tilde a(s)|\theta(s)|^2\,ds\,dt + \frac{1}{4}\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt.
\end{aligned}$$

Inserting the above bounds in (4.13) implies that

$$\begin{aligned}
&\int_0^{\tilde t_n}\tilde a(t)|\theta(t)|^2\,dt\\
&\qquad\le C\int_0^{\tilde t_n}|\zeta(t)|^2\,dt + 4\frac{\tilde t_n^\alpha}{\alpha}\left(\frac{\mu^*}{\mu_*}\right)^2\sum_{j=1}^n\int_{\tilde t_{j-1}}^{\tilde t_j}(\tilde t_n - t)^{\alpha-1}\,dt\int_0^{\tilde t_j}\tilde a(t)|\theta(t)|^2\,dt.
\end{aligned}$$

Therefore, the desired result now immediately follows after applying of the Gronwall inequality in Lemma 3.3 and using the assumption (1.4) on the function $\tilde a$ (instead of a). ∎

5. Superconvergence results

In this section, we study the nodal error analysis of the DG solution U defined by (2.3) with $U_-^0 = u_0$. We derive error estimate of the DG solution, giving rise to superconvergence algebraic rates. Our analysis partially relies on the techniques introduced in [24, Chapter 12] for parabolic problems.

Theorem 5.1. *Let $\alpha \in (0,1)$ in (1.3). Let the solution u of problem* (1.1) *satisfy the regularity property* (2.7) *and let $U \in \mathcal{W}_p$ be the DG approximate solution defined*

by (2.3) *with* $p \geq 1$. *In addition to the mesh assumption* (2.8) *and* (2.9), *we assume that* $k_n \geq k_{n-1}$ *for* $1 \leq n \leq N$. *Then*

- *for* $p = 1$,

$$\max_{1\leq n\leq N}|U_-^n - u(t_n)| \leq Ck \times \begin{cases} k^{\gamma\sigma}, & 1 \leq \gamma \leq 2/\sigma \\ k^2, & \gamma \geq 2/\sigma \end{cases}$$

- *and for* $p \geq 2$, *we have*

$$\max_{1\leq n\leq N}|U_-^n - u(t_n)| \leq C\max\{1, \log n\}k^{\alpha+1} \times \begin{cases} k^{\gamma\sigma}, & 1 \leq \gamma \leq (p+1)/\sigma \\ k^{p+1}, & \gamma \geq (p+1)/\sigma\,. \end{cases}$$

Proof. From (4.3), (4.2), (2.6) and (4.4) (recall that $\eta = \Pi^- u - u$), we observe that

$$\begin{aligned} (U_-^N - u(t_N))z_T &= G_N(U, Z) - G_N(u, z) \\ &= G_N(u, Z - z) = G_N(\eta, z - Z). \end{aligned} \tag{5.1}$$

The alternative expression for G_N given in Remark 2.2 and the equality $\eta(t_-^n) = 0$ show that

$$G_N(\eta, z - Z) = \delta_{1N} + \delta_{2N}, \tag{5.2}$$

where

$$\delta_{1N} = -\sum_{j=1}^{N}\int_{t_{n-1}}^{t_n} \eta\,(z - Z)'\,dt \quad \text{and} \quad \delta_{2N} = \int_0^{t_N} (a(t)\eta(t) + \mathcal{B}\eta(t))\,(z - Z)(t)\,dt.$$

To bound δ_{1N} and δ_{2N}, we start from the regularity property (4.10) and the relation $z(t) = \tilde{z}(t_N - t)$, and get

$$|z'(t)| + (t_N - t)^{1-\alpha}|z''(t)| + (t_N - t)^{2-\alpha}|z'''(t)| \leq C|z_T|\,. \tag{5.3}$$

For $p = 1$, the orthogonality property of Π^- yields

$$\begin{aligned} -\,\delta_{1N} = \sum_{j=1}^{N}\int_{t_{n-1}}^{t_n} \eta(t)\,z'(t)\,dt &= \sum_{j=1}^{N}\int_{t_{n-1}}^{t_n} \eta(t)\,[z'(t) - z'(t_n)]\,dt \\ &= \sum_{j=1}^{N}\int_{t_{n-1}}^{t_n} \eta(t)\int_t^{t_n} z''(s)\,ds\,dt \end{aligned}$$

and hence, with the help of (5.3) we have

$$\begin{aligned} |\delta_{1N}| &\leq C\|\eta\|_{J_N}\sum_{n=1}^{N} k_n\int_{t_{n-1}}^{t_n} |z''(t)|\,dt \\ &\leq Ck\|\eta\|_{J_N}\int_0^{t_N} (t_N - t)^{\alpha-1}|z_T|\,dt = Ck\|\eta\|_{J_N}|z_T|\,t_N^\alpha/\alpha\,. \end{aligned} \tag{5.4}$$

For $p \geq 2$, again the orthogonality property of Π^- gives

$$\begin{aligned} \delta_{1N} = -\sum_{j=1}^{N}\int_{t_{n-1}}^{t_n} \eta(t)\,z'(t)\,dt &= \sum_{j=1}^{N-1}\int_{t_{n-1}}^{t_n} \eta(t)\,[\Pi^+ z'(t) - z'(t)]\,dt \\ &\quad + \int_{t_{N-1}}^{t_N} \eta(t)\int_t^{t_n} z''(s)\,ds\,dt \end{aligned}$$

where $\Pi^+ z'$ is the discontinuous, piecewise-linear interpolant of z' defined by

$$\Pi^+ z'(t) := z'(t_{n-1}) + \frac{\bar{z}^n - z'(t_{n-1})}{k_n/2}(t - t_{n-1}) \quad \text{for } t \in I_n$$

with $\bar{z}^n := k_n^{-1} \int_{I_n} z'(t)\, dt$ denote the mean value of z' over the subinterval I_n.

Elementary calculations show that, for $t \in I_n$, the interpolation error has the representation

$$\Pi^+ z'(t) - z'(t) = \int_{t_{n-1}}^{t} (s-t) z'''(s)\, ds + \frac{t - t_{n-1}}{k_n^2} \int_{I_n} (t_n - s)^2 z'''(s)\, ds,$$

and so, by (5.3),

$$\begin{aligned}
|\delta_{1N}| &\le C\|\eta\|_{J_N} \left(\sum_{n=1}^{N-1} k_n^2 \int_{t_{n-1}}^{t_n} |z'''(t)|\, dt + C k_N \int_{t_{N-1}}^{t_N} |z''(t)|\, dt \right) \\
&\le C\|\eta\|_{J_N} |z_T| \left(\sum_{n=1}^{N-1} k_n^2 \int_{t_{n-1}}^{t_n} (t_N - t)^{\alpha-2}\, dt + k_N \int_{t_{N-1}}^{t_N} (t_N - t)^{\alpha-1}\, dt \right) \\
&\le C\|\eta\|_{J_N} |z_T| (k^{\alpha+1} \log(t_N/k_N) + k_N^{\alpha+1})
\end{aligned}$$

where in the last step we used; $k_n \ge k_{n-1}$ for $n \ge 1$, and

$$\begin{aligned}
&\sum_{n=1}^{N-1} k_n^2 \int_{t_{n-1}}^{t_n} (t_N - t)^{\alpha-2}\, dt \\
&\qquad\qquad \le C \sum_{n=1}^{N-1} k_n^{1+\alpha} \int_{t_{n-1}}^{t_n} (t_N - t)^{-1}\, dt \le C k^{1+\alpha} \int_0^{t_{N-1}} (t_N - t)^{-1}\, dt.
\end{aligned}$$

To bound δ_{2N}, we use (1.4), integrating, applying the Holder's inequality and then, using Theorem 4.1, we notice that

$$\begin{aligned}
|\delta_{2N}| &\le \mu^* \|\eta\|_{J_N} \int_0^{t_N} \left(|Z(t) - z(t)| + \int_0^t (t-s)^{\alpha-1}\, ds\, |Z(t) - z(t)| \right) dt \\
&\le \mu^* \|\eta\|_{J_N} (1 + t_N^\alpha/\alpha) \int_0^{t_N} |Z(t) - z(t)|\, dt \\
&\le C\|\eta\|_{J_N} \left(\int_0^{t_N} |Z(t) - z(t)|^2\, dt \right)^{1/2} \le C k^{\alpha+1} \|\eta\|_{J_N} |z_T|\, .
\end{aligned}$$

Using Theorem 3.1, the regularity assumption (2.7), and the mesh assumption (2.9), we get

$$\|\eta\|_{I_1}^2 \le C k_1 \int_0^{t_1} |u'(t)|^2\, dt \le C k_1 \int_0^{t_1} t^{2\sigma-2}\, dt = C \frac{t_1^{2\sigma}}{2\sigma - 1} \le C k^{2\gamma\sigma},$$

and for $n \ge 2$, we use (2.8) instead of (2.9) and obtain

$$\begin{aligned}
\|\eta\|_{I_n}^2 &\le C k_n^{2p+1} \int_{t_{n-1}}^{t_n} |u^{(p+1)}(t)|^2\, dt \\
&\le C k_n^{2p+1} \int_{t_{n-1}}^{t_n} t^{2\sigma-2p-2}\, dt \le C\, k_n^{2p+2} t_n^{2\sigma-2p-2} \le C\, k^{2p+2} t_n^{2\sigma-(2p+2)/\gamma}.
\end{aligned}$$

Finally, combine the above estimations from δ_{1N} and δ_{2N}, and recalling (5.2) and (5.1) yield the desired bound for $n = N$. For the nodal error at any time step t_{n_0}

with $1 \le n_0 \le N$, we follow the above steps with n_0 in place of N, which will then complete the proof. $\square$

6. Super-convergence analysis for smooth kernels

In this section, we handle the nodal super-convergence error analysis of the DG scheme (2.3) for problem (1.1) when $\alpha \in \mathbb{N}_0$ (so the kernel is smooth). We use a uniform mesh with step-size $k = T/N$ where k is assumed to be sufficiently small. In our analysis, we follow the derivations of Sections 4 and 5 with some modifications. We assume that the functions a, b and f are sufficiently regular such that the solution u of (1.1) satisfies $|u^{(j)}(t)| \le C$ (and consequently $|\tilde{z}^{(j)}(t)| \le C|z_T|$) for $1 \le j \le p+1$ with $t \in (0,T]$. Thus, from Theorem 3.1 we notice that for $n \ge 1$,

$$\|\eta\|_{I_n}^2 \le Ck^{2p+1}\int_{t_{n-1}}^{t_n} |u^{(p+1)}(t)|^2\,dt \le Ck^{2p+2}. \tag{6.1}$$

We start our analysis by deriving the error involved in approximating the solution z of the backward VIE (4.1).

Theorem 6.1. *If z is the solution of the backward VIE* (4.1)*, and if $Z \in \mathcal{W}_p$ is the approximate solution defined by* (4.3)*, then*

$$\int_0^{t_N} |z - Z|^2\,dt \le Ck^{2p+2}|z_T|^2\,.$$

Proof. First, we recall (4.13) (over a uniform mesh)

$$\frac{3}{4}\int_0^{t_n} \tilde{a}(t)|\theta(t)|^2\,dt \le \mu^*\int_0^{t_n} |\zeta(t)|^2\,dt + \int_0^{t_n} |\tilde{\mathcal{B}}\zeta(t) + \tilde{\mathcal{B}}\theta(t)|\,|\theta(t)|dt. \tag{6.2}$$

Using the fact that $\alpha - 1 \ge 0$, and the Cauchy-Schwarz and the geometric-arithmetic mean inequalities, we observe

$$\begin{aligned}
\int_0^{t_n} |\tilde{\mathcal{B}}\zeta(t)\,\theta(t)|dt &\le \mu^*\int_0^{t_n} t^{\alpha-1}|\theta(t)|\int_0^t |\zeta(s)|\,ds\,dt \\
&\le \frac{\mu^*}{\mu_*}\int_0^{t_n} t^{\alpha-1}\tilde{a}(t)^{1/2}|\theta(t)|\int_0^t \tilde{a}(s)^{1/2}|\zeta(s)|\,ds\,dt \\
&\le \frac{\mu^*}{\mu_*}\int_0^{t_n} t^{\alpha-\frac{1}{2}}\,\tilde{a}(t)^{1/2}|\theta(t)|\left(\int_0^t \tilde{a}(s)|\zeta(s)|^2\,ds\right)^{1/2} dt \\
&\le \left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{t_n} t^{2\alpha-1}\int_0^t \tilde{a}(s)|\zeta(s)|^2\,ds\,dt + \frac{1}{4}\int_0^{t_n} \tilde{a}(t)|\theta(t)|^2\,dt \\
&\le \frac{t_n^{2\alpha}}{2\alpha}\left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{t_n} \tilde{a}(s)|\zeta(s)|^2\,ds + \frac{1}{4}\int_0^{t_n} \tilde{a}(t)|\theta(t)|^2\,dt\,.
\end{aligned}$$

Similarly, we notice that

$$\begin{aligned}
&\int_0^{t_n} |\tilde{\mathcal{B}}\theta(t)\,\theta(t)|dt \\
&\qquad\le \left(\frac{\mu^*}{\mu_*}\right)^2\int_0^{t_n} t^{2\alpha-1}\int_0^t \tilde{a}(s)|\theta(s)|^2\,ds\,dt + \frac{1}{4}\int_0^{t_n} \tilde{a}(t)|\theta(t)|^2\,dt.
\end{aligned}$$

Inserting the above bounds in (6.2) yields

$$\int_0^{t_n} \tilde{a}(t)|\theta(t)|^2\,dt \\ \leq C\int_0^{t_n} |\zeta(t)|^2\,dt + t_n^\alpha \left(\frac{\mu^*}{\mu_*}\right)^2 \sum_{j=1}^n \int_{t_{j-1}}^{t_j} t^{\alpha-1}\,dt \int_0^{t_j} \tilde{a}(s)|\theta(s)|^2\,ds.$$

Since one can show by induction on α ($\alpha \in \mathbb{N}_0$) that

$$\int_{t_{j-1}}^{t_j} t^{\alpha-1}\,dt = \frac{1}{\alpha}[t_j^\alpha - t_{j-1}^\alpha] = \frac{k^\alpha}{\alpha}[j^\alpha - (j-1)^\alpha] \leq k^\alpha j^{\alpha-1} \leq k,$$

an application of the standard discrete Gronwall Lemma gives

$$\int_0^{t_n} |\theta(t)|^2\,dt \leq C\int_0^{t_n} |\zeta(t)|^2\,dt \quad \text{for } 1 \leq n \leq N.$$

Hence, (4.9) is valid now and therefore, we obtain the desired result after noting that

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\zeta(t)|^2\,dt \leq C\sum_{n=1}^N k^{2p+2} \int_{t_{n-1}}^{t_n} |\tilde{z}^{(p+1)}(t)|^2\,dt \leq Ck^{2p+2}|z_T|^2\,.$$

□

In the next theorem we study the nodal error analysis of the DG solution U defined by (2.3) with $U_-^0 = u_0$.

Theorem 6.2. *Let $\alpha \in \mathbb{N}_0$ in (1.3). Let the solution u of problem* (1.1) *be sufficiently regular and let $U \in \mathcal{W}_p$ be the DG approximate solution defined by* (2.3) *with $p \geq 1$. Then we have*

$$\max_{1\leq n\leq N} |U_-^n - u(t_n)| \leq Ck^{2p+1}\,.$$

Proof. We follow the steps given in the proof of Theorem 5.1, however we use the new bounds of δ_{1N} and δ_{2N} derived below. The orthogonality property of Π^- gives

$$\delta_{1N} = -\sum_{j=1}^N \int_{t_{n-1}}^{t_n} \eta(t)\,z'(t)\,dt = \sum_{j=1}^N \int_{t_{n-1}}^{t_n} \eta(t)\,[\widehat{\Pi} z'(t) - z'(t)]\,dt$$

where $\widehat{\Pi} z' \in \mathcal{W}_{p-1}$ is defined by: for $1 \leq n \leq N$,

$$\widehat{\Pi} z'(t_n^-) = z'(t_n) \text{ and } \int_{t_{n-1}}^{t_n} (z' - \widehat{\Pi} z')\,v\,dt = 0 \quad \forall\, v \in \mathbb{P}_{p-2}(I_n)\,.$$

Hence, from Theorem 3.1, there exists a constant C, which depends on p such that:

$$\|\widehat{\Pi} z' - z'\|_{I_n}^2 \leq Ck^{2p-1} \int_{t_{n-1}}^{t_n} |z^{(p+1)}|^2\,dt \leq Ck^{2p}|z_T|^2 \quad \text{for } 1 \leq n \leq N$$

and thus, using (6.1) we obtain

$$|\delta_{1N}| \leq Ck\sum_{n=1}^N \|\eta\|_{I_n} \|\widehat{\Pi} z'(t) - z'(t)\|_{I_n} \leq Ck^{2p+1}|z_T|\,.$$

For the bound of δ_{2N}, we use (1.4), integrating, applying the Cauchy-Schwarz inequality and then, using (6.1) and Theorem 6.1, we notice that

$$\begin{aligned}|\delta_{2N}| &\le \mu^*\|\eta\|_{J_N}\int_0^{t_N}\left(|Z(t)-z(t)|+\int_0^t (t-s)^{\alpha-1}\,ds\,|Z(t)-z(t)|\right)dt\\ &\le \mu^*\|\eta\|_{J_N}(1+t_N^\alpha/\alpha)\int_0^{t_N}|Z(t)-z(t)|\,dt\\ &\le C\|\eta\|_{J_N}\left(\int_0^{t_N}|Z(t)-z(t)|^2\,dt\right)^{1/2}\le Ck^{2p+2}|z_T|\,.\end{aligned}$$

□

7. Numerical examples

In this section, we present a set of numerical experiments to demonstrate the obtained theoretical error estimates and also, to justify the validation of the DG scheme (2.3) for a wider class of integro-differential equations.

Throughout, we consider problem (1.1) with $T=1$, the initial data $u_0=0$ and $b(t)=1/\Gamma(\alpha)$ (Here, Γ denotes the usual gamma function.). Recall that, u denotes the exact solution of (1.1) and U is the DG solution defined by (2.3) using 2^i $(i\ge 1)$ subintervals, that is $N=2^i$.

7.1. **Example 1.** Choosing the coefficient $a(t)$ and the source term $f(t)$ such that the solution u of (1.1) is given by

$$u(t)=t^{\alpha+1}\mathrm{e}^{-t}. \tag{7.1}$$

For $\alpha\in(0,1)$, we notice that near $t=0$, u'' is not bounded, however u is smooth away from $t=0$. So, we employ a time mesh of the form (2.10) for various choices of the mesh grading parameter $\gamma\ge 1$ to verify the results of Theorem 5.1.

Since the exact solution (7.1) behaves like $t^{\alpha+1}$ as $t\to 0^+$, we see that the regularity condition (2.7) holds for $\sigma=\alpha+1$. Thus, from Theorem 5.1 and by ignoring the logarithmic factor, we expect

$$\begin{aligned}\|U-u\|_{node} &:= \max_{1\le n\le N}|U_-^N-u(t_n)|\\ &= \begin{cases} O(k^{\gamma(\alpha+1)+\min\{p,\alpha+1\}}) & \text{for } 1\le\gamma<(p+1)/(\alpha+1),\\ O(k^{p+1+\min\{p,\alpha+1\}}) & \text{for } \gamma\ge(p+1)/(\alpha+1).\end{cases}\end{aligned}$$

Case 1 Choosing $a(t)=1$, thus

$$f(t)=(\alpha+1)t^\alpha\mathrm{e}^{-t}+t^{2\alpha+1}\sum_{i=0}^{\infty}(-1)^i\frac{t^i}{i!}\frac{\Gamma(2+\alpha+i)}{\Gamma(2+2\alpha+i)}\,. \tag{7.2}$$

To illustrate the theoretical results of Theorem 6.2, we choose $\alpha=2$ and so, the memory term and the solution u are smooth. As expected, the numerical results in Table 1 demonstrate nodal errors of order $O(k^{2p+1})$ for $p=1,2,3$.

In Tables 2–4 we displayed the nodal error $\|U-u\|_{node}$ over the mesh (2.10) with $N=2^i$ and for different values of γ when $\alpha=0.2$ and $\alpha=0.5$ (So $|u^{(j)}(t)|$ is not bounded near $t=0$ for $j\ge 2$.). Results shown in these tables confirm that the best convergence rate we can achieve is $O(k^{p+1+\min\{p,\alpha+1\}})$ and thus our theoretical results in Theorem 5.1 are sharp in terms of the convergence order. However,

i	$p=1$		$p=2$		$p=3$	
2	3.953e-05	2.622	1.675e-07	4.934	1.928e-10	6.949
3	5.430e-06	2.864	5.391e-09	4.958	1.537e-12	6.971
4	7.063e-07	2.943	1.712e-10	4.976	1.199e-14	7.002
5	8.991e-08	2.974	5.409e-12	4.985	1.003e-16	6.902

TABLE 1. The nodal error and the convergence rate over a uniform mesh with $N = 2^i$ subintervals when $\alpha = 2$ in (7.1)–(7.2).

i	$\gamma=1$		$\gamma=1.25$		$\gamma=1.4$	
6	6.839e-08	1.886	3.991e-08	3.076	4.883e-08	3.085
7	1.522e-08	2.168	4.746e-09	3.071	5.767e-09	3.082
8	3.118e-09	2.286	5.668e-10	3.066	6.836e-10	3.076
9	6.147e-10	2.342	6.796e-11	3.060	8.136e-11	3.070

TABLE 2. The nodal error and the rate of convergence for Case 1 when $\alpha = 0.2$ and $p = 1$.

	i	$\gamma=1$		$\gamma=4/3$		$\gamma=(p+2.5)/3$	
$p=2$	6	5.19e-10	3.08	8.23e-12	4.09	4.43e-12	4.45
	7	6.28e-11	3.04	5.01e-13	4.04	2.00e-13	4.46
	8	7.73e-12	3.02	3.10e-14	4.01	9.01e-15	4.47
$p=3$	4	2.36e-09	3.13	1.40e-10	4.07	1.80e-11	5.47
	5	2.83e-10	3.06	8.60e-12	4.03	4.01e-13	5.48
	6	3.47e-11	3.03	5.33e013	4.01	8.90e-15	5.49

TABLE 3. The nodal error and the convergence rate for Case 1 when $\alpha = 0.5$ and $p = 2\,,3$.

it indicates that in practice we can relax the restriction on the mesh grading exponent γ. We conjecture that $\gamma \geq (p+1+\min\{p,\alpha+1\})/(\sigma+\alpha+1)$ suffices to ensure $O(k^{p+1+\min\{p,\alpha+1\}})$ convergence. More precisely, we observe $O(k^{(\sigma+\alpha+1)\gamma})$-rates if $1 \leq \gamma \leq (p+1+\min\{p,\alpha+1\})/(\sigma+\alpha+1)$. In Table 2, we have chosen $\alpha = 0.2$ in (7.1)–(7.2) and the DG solution $U \in \mathcal{W}_1$ (i.e., the approximate solution is a piecewise linear polynomial). An $O(k^{(\sigma+\alpha+1)\gamma})$ (i.e., $O(k^{2.4\gamma})$) convergence rate has been observed if $1 \leq \gamma < 3/(\sigma+\alpha+1)$ and $O(k^3)$ if $\gamma \geq 3/(\sigma+\alpha+1)$. In Table 3, we considered $\alpha = 0.5$ and $U \in \mathcal{W}_p$ where $p = 2$ or 3. An $O(k^{(\sigma+\alpha+1)\gamma})$ convergence rate has been demonstrated if $1 \leq \gamma \leq (p+2+\alpha)/(\sigma+\alpha+1)$. Finally, we chose $\gamma > (p+2+\alpha)/(\sigma+\alpha+1)$ in Table 4 and we realized that the order of convergence almost matched the one given in the last column of Table 3 where $\gamma = (p+2+\alpha)/(\sigma+\alpha+1)$ (i.e., the order of convergence did not exceed $p+2+\alpha$ for $p \geq 2$ as the theoretical results suggested).

Case 2 Choosing $a(t) = t^\alpha + 1$ and so

$$f(t) = (\alpha+1)t^\alpha e^{-t} + t^{2\alpha+1}e^{-t} + t^{2\alpha+1}\sum_{i=0}^{\infty}(-1)^i \frac{t^i}{i!}\frac{\Gamma(2+\alpha+i)}{\Gamma(2+2\alpha+i)}\,. \tag{7.3}$$

	i	Error	Rate
$p=2$, $\gamma=5/3$	6	4.6578e-12	4.4511
	7	2.1039e-13	4.4685
	8	9.3953e-15	4.4850
$p=3$, $\gamma=2$	3	1.1677e-09	5.4859
	4	2.5328e-11	5.5269
	5	5.6022e-13	5.4986

TABLE 4. Nodal errors and convergence rates for Case 1 when $\alpha=0.5$ and $p=2\,,3$.

i	$\gamma=1$		$\gamma=1.25$		$\gamma=1.4$	
6	1.633e-07	2.305	9.644e-08	3.024	1.233e-07	3.024
7	3.205e-08	2.350	1.184e-08	3.026	1.514e-08	3.026
8	6.220e-09	2.365	1.454e-09	3.026	1.858e-09	3.026
9	1.205e-09	2.367	1.787e-10	3.024	2.284e-10	3.024

TABLE 5. The nodal error and the rate of convergence for Case 2 when $\alpha=0.2$ and $p=1$.

	i	$\gamma=1$		$\gamma=4/3$		$\gamma=(p+2.5)/3$	
$p=2$	6	1.55e-09	3.01	2.62e-11	4.01	6.94e-12	4.40
	7	1.92e-10	3.01	1.63e-12	4.01	3.21e-13	4.43
	8	2.39e-11	3.01	1.02e-13	4.00	1.47e-14	4.45
$p=3$	4	5.90e-09	2.62	3.76e-10	3.68	1.63e-11	5.49
	5	8.22e-10	2.84	2.60e-11	3.85	3.59e-13	5.50
	6	1.08e-10	2.93	1.71e-12	3.93	8.07e-15	5.47

TABLE 6. The nodal error and the convergence rate for Case 2 when $\alpha=0.5$ and $p=2\,,3$..

In Tables 5 and 6 we displayed the nodal error $\|U-u\|_{node}$ over the mesh (2.10) with $N=2^i$ and for different values of γ. Again, we observe convergence of order $O(k^{(\sigma+\alpha+1)\gamma})$ if $1\le\gamma<(p+1+\min\{p,\alpha+1\})/(\sigma+\alpha+1)$ and of order $O(k^{p+1+\min\{p,\alpha+1\}})$ if $\gamma\ge(p+1+\min\{p,\alpha+1\})/(\sigma+\alpha+1)$ for different polynomial degrees p.

7.2. **Example 2.** In this example we demonstrate that the nodal superconvergence results of Theorem 5.1 are still valid even if $a(t)\equiv 0$ in (1.1) (so the assumption (1.4) is not satisfied) with $\alpha\in(0,1)$.

In this case, (1.1) reduces to the following (scalar evolution or fractional wave equation, see [11, 13]) time-dependent problem: for $\alpha\in(0,1)$,

$$u'+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds=f(t)\quad\text{for } 0<t<T \text{ with } u(0)=0\,. \tag{7.4}$$

The piecewise linear ($p=1$) DG method for (7.4) had been studied extensively in [13]. However, for $p\ge2$, the stability and convergence analyses of the DG method for (7.4) are more difficult and it will be a topic of future research.

i	$\gamma = 1$		$\gamma = 1.25$		$\gamma = 1.5$	
6	9.11e-08	2.33	1.70e-08	2.79	2.59e-08	2.76
7	1.76e-08	2.37	2.38e-09	2.84	3.67e-09	2.82
8	3.37e-09	2.39	3.24e-10	2.87	5.05e-10	2.86
9	6.40e-10	2.39	4.34e-11	2.90	6.83e-11	2.89

Table 7. The nodal error and the convergence rates for Example 2, when $\alpha = 0.2$ and $p = 1$.

	i	$\gamma = 1$		$\gamma = 4/3$		$\gamma = (p + 2.5)/3$	
$p = 2$	5	3.94e-09	3.03	1.27e-10	4.32	1.77e-10	4.47
	6	4.89e-10	3.01	7.89e-12	4.01	7.89e-12	4.48
	7	6.09e-11	3.00	4.92e-13	4.00	3.51e-13	4.49
$p = 3$	4	2.17e-09	3.00	1.36e-10	4.00	2.23e-11	5.35
	5	2.72e-10	3.00	8.49e-12	4.00	5.74e-13	5.28
	6	3.40e-11	3.00	5.31e013	4.00	2.50e-14	5.12

Table 8. The nodal error and the convergence rate for Example 2 when $\alpha = 0.5$ and $p = 2\,,3$.

Using the Mittag–Leffler function $E_\mu(x) = \sum_{p=0}^{\infty} x^p/\Gamma(1 + p\mu)$, we may write the exact solution as

$$u(t) = \int_0^t E_{\alpha+1}(-s^{\alpha+1}) f(t-s)\, ds\,.$$

Choosing a source term $f(t) = (\alpha + 1)t^\alpha$, we find that

$$u(t) = -\Gamma(\alpha+2) \sum_{p=1}^{\infty} \frac{(-t)^{(\alpha+1)p}}{\Gamma(1 + (\alpha+1)p)} = \Gamma(\alpha+2)\left(1 - E_{\alpha+1}(-t^{\alpha+1})\right). \tag{7.5}$$

Since the exact solution of (7.4) behaves like $t^{\alpha+1}$ as $t \to 0^+$, we see that the regularity conditions (2.7) hold for any $\sigma = \alpha + 1$. For $p = 1$ (that is, piecewise linear DG method), the numerical results shown in Table 7 demonstrate a nodal superconvergence rate of order $O(k^{\gamma(\sigma+\alpha+1)})$ for $1 \le \gamma < 3/(\sigma + \alpha + 1)$, and of order $O(k^3)$ for $\gamma \ge 3/(\sigma + \alpha + 1)$. However, for $p \ge 2$, the numerical results shown in Table 8 illustrated a nodal error estimates of order $O(k^{\gamma(p+2+\alpha})$ (that is, $O(k^{\gamma(\sigma+p+1)}))$ for $1 \le \gamma < (p + 2 + \alpha)/(\sigma + \alpha + 1)$, and almost of order $O(k^{p+2+\alpha})$ for $\gamma \ge (p + 2 + \alpha)/(\sigma + \alpha + 1)$.

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